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# Realizable Configurations of Lines in Pictures of Polyhedra†

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## INTRODUCTION

In an idealized picture of a scene that contains only polyhedra each line segment that is recorded can have only one of four possible "meanings". In order to understand the picture it is necessary that we be able to label each line with one of the four corresponding labels: +, -, → or ←. A "+" or "-" label is associated, respectively, with a convex or concave edge that has both of its two associated planes visible. A line labelled with an arrow refers to a convex edge oriented so that only one of these two planes is visible from the camera and the other is hidden behind it. The orientation of the arrow along the line is such that the planes are to the right of the arrow. If no consistent set of line labels is possible the picture is of an "impossible" object. If one or more labellings are possible a necessary condition for the picture to be realizable will have been satisfied and the picture may, indeed, be ambiguous. An earlier paper (Huffman, 1971) dealt with the restricted case of scenes containing only trihedral objects. This paper generalizes the results of the earlier one to the case of scenes that contain polyhedra having arbitrary numbers of planes associated with the vertices.

The catalog of the twelve possible pictures of trihedral vertices is shown in Figure 1. When polyhedra with arbitrary numbers of edges incident at each vertex are considered the development of an extended catalog is clearly impossible; it would need to contain an infinite number of entries. What is needed instead is a decision procedure that can be applied to any configuration of labelled lines that is a candidate for inclusion in that hypothetical catalog.

We shall also generalize this procedure still further so that it applies not only to the configurations of lines incident at a single picture node, but also to the situation in which an arbitrary number of lines having arbitrary labels enter a

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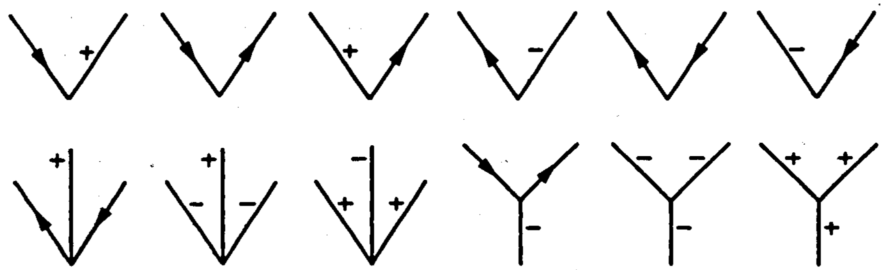


FIG. 1. Catalog of labelled-line configurations possible at nodes in pictures of trihedral objects.

closed picture region from arbitrary directions. The procedure also indicates, if the configuration is realizable, how the planes bounded by these lines can be oriented.

A by-product of the effort required in developing the main theme is a list of all possible interpretations of picture configurations of the form "T". For the trihedral case such a configuration could not correspond to a physical vertex and could only be interpreted as evidence that one edge was obscured by another.

In the research reported here use is made of the concept of the *dual-scene* and the *dual-picture* reported in another paper in this volume. Briefly, the entity dual to plane $z = ax + by + c$ in the scene is defined to be the point at $(u,v,w) = (a,b,-c)$ in the dual-scene. The corresponding projected point in the dual-picture is at $(u,v) = (a,b)$. The intersection of two planes in the scene determines a line that has a correspondent in the dual-scene. This latter line can be thought of as comprised of an infinite set of points each of which represents one of the set of planes containing the given line in the scene.

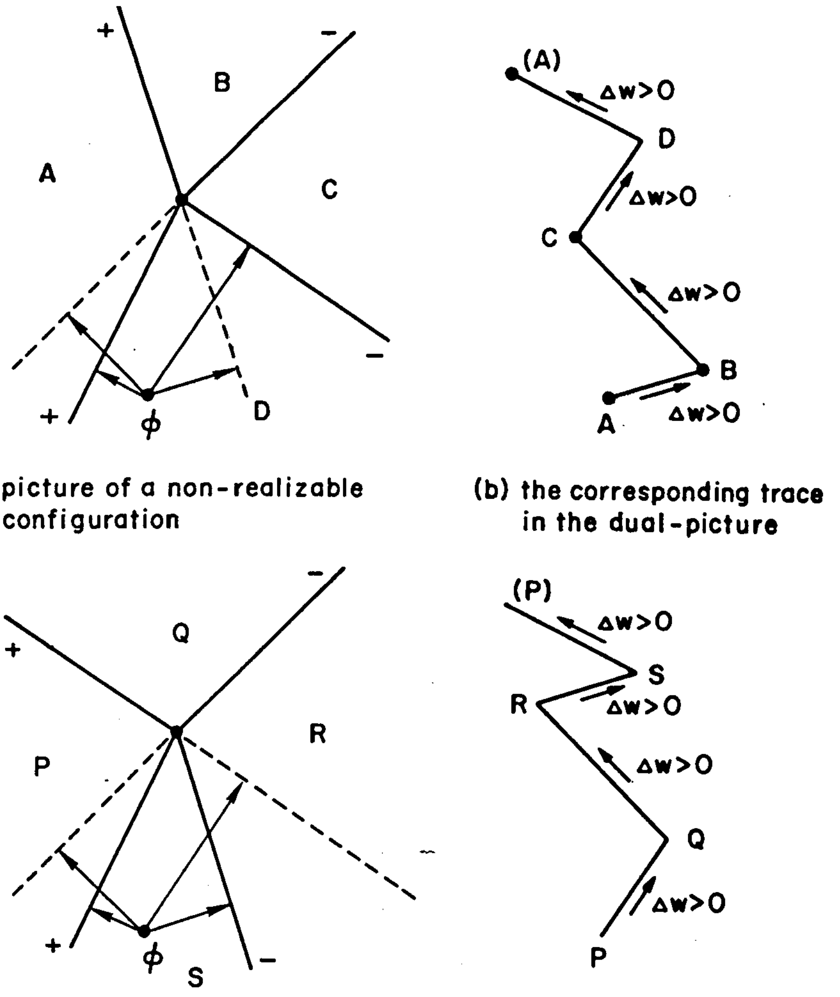
A useful property of such a dual pair of lines is that the rate of change of z along a given scene line with respect to motion along the projected line in the (x,y) picture plane is equal to the distance from the origin in the (u,v) dual-picture plane to the corresponding dual-scene line that is projected upon it. Similarly, the rate of change of w along a given dual-scene line with respect to motion along the projected dual-line in the (u,v) dual-picture plane is equal to the distance from the origin in the (x,y) picture plane to the scene line that is projected upon it.

A most delightful consequence of this definition of dual is that a dual pair of lines (in the scene and dual-scene) are recorded in the picture and dual-picture by lines that are at right angles to each other.

Finally, a point in the scene corresponds in the dual-scene to the plane of all points dual to planes containing the given point of the scene. Specifically, the point at $(x,y,z) = (d,e,f)$ corresponds to the plane $w = du + ev - f$.

## CONFIGURATIONS OF LINES INCIDENT AT A SINGLE PICTURE NODE

As we move along any path on the surface of a polyhedron there is generated




(a) picture of a non-realizable configuration

(b) the corresponding trace in the dual-picture

(c) picture of a non-realizable configuration

(d) the corresponding trace in the dual-picture

FIG. 2. A pair of equivalent non-realizable configurations and their dual–pictures.

a corresponding path in the dual-scene (and in the associated dual-picture). We shall call this latter path the *trace* of the original path. The trace will consist of an ordered sequence of points (associated with faces of the polyhedron) joined by lines (associated with edges of the polyhedron). We shall be interested in this paper in traces associated with closed contours drawn on the polyhedral faces in the scene. Such a closed contour may enclose just one polyhedral vertex, or it may enclose more than one. In either case since a contour is closed the corresponding trace must also be closed (if the polyhedron is to be realizable). Note, for example, that a trace may appear to close in the dual-picture but not

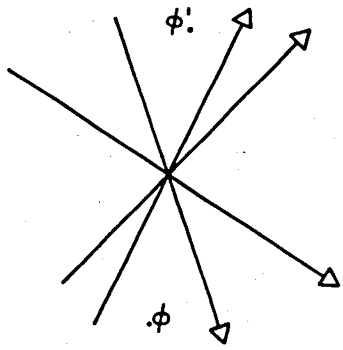



FIG. 3. An equivalent representation of the configurations of Figure 2.

actually close in the dual-scene. For instance, if a trace starts at a point (u,v,w) = (a,b,c) but ends at (u,v,w) = (a,b,c′) then the scene path begins and ends not on the same plane but on two different (parallel) planes.

A configuration of labelled lines crossed by a contour in a picture will be termed *realizable* if and only if it can be demonstrated that the corresponding trace can be closed (in the dual-scene). In other words, a configuration of picture lines is realizable when and only when there exists at least one orientation of each of the associated planes such that their intersections would yield the lines of the given configuration.

We shall first deal exclusively with the issue of the realizability of configurations of lines that are all labelled "+" or "-". It will then be easy to modify our results so that they apply to configurations that contain arrow-labelled lines.

Consider first the nodal configuration shown in Figure 2-a. The lengths of the line segments of the trace of Figure 2-b are not determined by the picture, but their directions are. It is apparent that the trace cannot be closed because each portion of the trace has a positive v-component. Hence the configuration is not realizable.

Alternatively, one can observe that if we were to choose the origin of the picture at the point $\phi$ the directions from there to each of the four lines are those shown. These same directions are those in which w increases along the associated segments of the trace (Huffman, 1976). Thus for that choice of origin, not only does the projected trace not close in the (u,v) plane, but neither can the net change of w along the trace be zero, as required.

The configuration of Figure 2-c is related to the example above in that certain of its lines are given the other sign-label and "*reversed*" (that is, extended on the opposite side of the given node). The resulting trace has components that take the earlier directions although they appear in a different order. The second configuration is thus also not realizable.

It is apparent that *any* of the configurations that result from reversing lines is unrealizable. The arrangement of *oriented lines* of Figure 2-e then can be used for all such configurations. The direction indicated by an open arrowhead is the

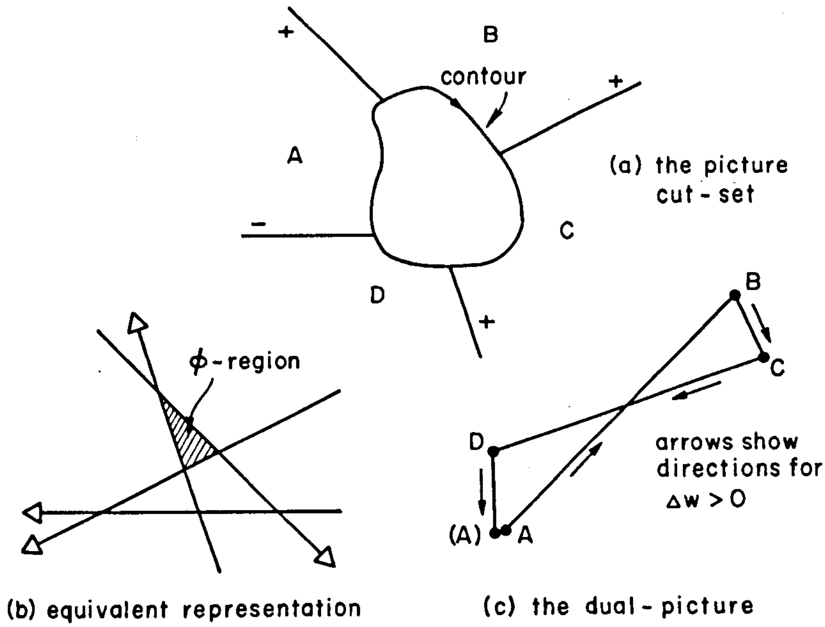
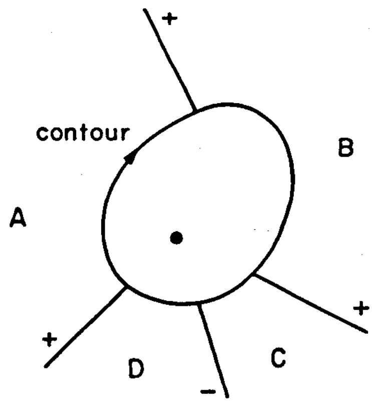



FIG. 4. An unrealizable cut-set.

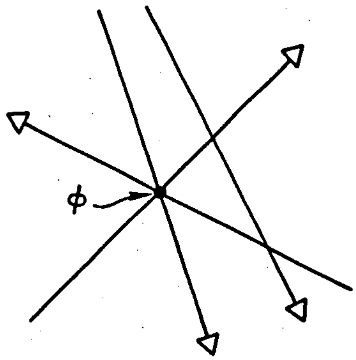
same as that that a "+" labelled line would take as it entered the region enclosed by the contour and opposite to that that a "-" labelled line would take. The direction in which w increases along the trace is obtained from that directed line by rotating it 90° counterclockwise. The fact that there exists a possible location $\phi$ that is to the right of all directed lines in the picture then guarantees that the net change in w along the trace would be positive, rather than zero. Similarly, the origin $\phi'$ is to the left of all directed lines and would lead to a net negative change in w. (Note that the directed line segments designated by *open* arrowheads are not the same as the line *labels* that have been used earlier.)

Still another method of understanding the non-realizable configuration of Figure 2-a is to imagine piercing the scene at the location $\phi$. If each of the four planes depicted were extended then along the ray $\phi$ from the viewer toward the scene the distance from the observer to the A-plane would be greater than to the B-plane (because their common edge is labelled "+" (convex) as shown). That is, $z_A > z_B$. Similarly $z_B > z_C$. Because the line common to planes C and D is concave we can conclude that at $\phi$ the distance to C is greater than that to D; that is, $z_C > z_D$. Similarly $z_D > z_A$. It is however *impossible* that $z_A > z_B > z_C > z_D > z_A$. Such a relationship is an example of what we shall call here a *cyclic inequality*. The existence of a point $\phi$ (or $\phi'$), at which a cyclic inequality exists, can be used to prove that the configuration is not realizable.

We see from the preceding examples that only the orientations of the directed lines in the equivalent configuration of Figure 3 are important in determining the realizability of the configuration(s) that it replaces.



(a) the picture cut-set



(b) equivalent representation

FIG. 5. A cut-set having a single $\phi$ - point.

## CONFIGURATIONS OF LINES NOT ALL INCIDENT AT A SINGLE PICTURE NODE

As an example of a contour around a region into which picture lines enter from more general directions consider Figure 4-a. (Obviously two or more nodes would be necessary inside the contour.) The equivalent oriented line representation is given in Figure 4-b. A region to the right of all lines is indicated. That region is thus the location for possible $\phi$ points any one of which could be used to demonstrate that the configuration is not realizable.

The dual-picture in Figure 4-c shows a trace that would seem at first glance to be closed. If we recall that the directions in which w increases can be found by turning the corresponding oriented lines 90° counterclockwise we discover that the trace only appears to close. Alternately we can demonstrate a cyclic inequality at any one of the $\phi$-points. Because we can now predict that the increments of w along a trace will be uniformly positive (or uniformly negative) when the existence of a $\phi$- (or $\phi'$-) point can be established we shall no longer bother in these cases to attempt to demonstrate candidates for closed traces.

We shall now give an example of a non-realizable configuration for which there is an isolated $\phi$-point. In Figure 5 a cut-set is demonstrated having the property that all but one of the four lines is incident at a single point. The equivalent picture of oriented lines indicates that a $\phi$-point (the only possible one) exists at the location common to three of the lines. At that $\phi$-point it can easily be shown that $z_A > z_B = z_C = z_D = z_A$. Obviously this is an impossibility.

It is now possible to give a general *definition of a $\phi$-point ($\phi'$-point)* based on representations using oriented lines:

> A $\phi$-point ($\phi'$-point) of an arbitrary cut-set of lines in a picture of polyhedra is a point that is to the right of (left of) some line of the cut-set and that is not to the left of (right of) any other lines.

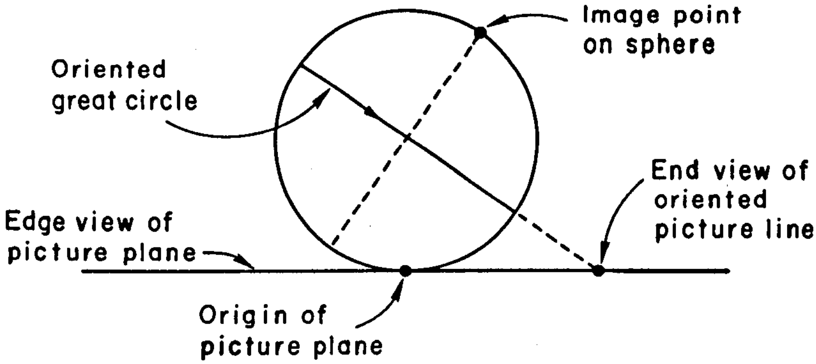



FIG. 6. Correspondence of image point to an oriented picture line.

We have demonstrated that a $\phi$- (or $\phi'$-) point implies that a cyclic inequality exists and thus that the cut-set of lines is not realizable. We shall prove below that a cut-set for which there is no $\phi$- (or $\phi'$-) point is realizable.

The major steps in our proof are the following: we will first show, for realizable cut-sets of +/- labelled lines entering a region of a picture in such a way that no three or more lines have a common intersection, that if there is no $\phi$- or $\phi'$-point there must exist some subset of four lines of the cut-set for which there is also no $\phi$- or $\phi'$-point. Secondly, we will show that this basic cut-set of four lines does have a realization. Finally, we will indicate that the other lines of the original cut-set can be added back to the cut-set without affecting the issue of realizability. It will be apparent in the development of the proof how to deal with the special case of a cut-set that has all of its lines incident at a single point.

## A "SPHERICAL" REPRESENTATION OF PICTURE LINES

During the first part of the proof we shall use an alternate representation of the oriented picture lines. Imagine (see Figure 6) a unit radius sphere tangent to the picture plane at its origin. An oriented picture line and the center of the sphere determine a plane that intersects the surface of the sphere in a great circle that has an orientation that corresponds to that of the oriented line. In Figure 6 we have assumed that the oriented line is pointed into the paper. This oriented great circle, in turn, determines (using a "right-hand rule") the direction of a unit *normal vector* and a corresponding uniquely determined *image point* on the sphere. It is a straightforward matter to show that lines that pass through a common picture point correspond to images that lie on a common great circle.

A fact that is especially important for our purposes is that if a set of picture lines all pass some point in the picture plane clockwise (or all pass counterclockwise) their images all lie in some common hemisphere. We use this fact to establish that when there exists a $\phi$- (or $\phi'$-) point for a cut-set of picture lines we can determine some corresponding great circle and hemisphere such that all normal vectors terminate either in the hemisphere or on the great circle boundary of that hemisphere. It is easy to show also that an appropriate boundary of that hemisphere is that great circle that is orthogonal to the line that contains the sphere center and the $\phi$- (or $\phi'$-) point of the picture plane.

We can see therefore that when and only when there is *no* $\phi$- or $\phi'$- point can positive linear combinations of the normal vectors that correspond to the picture lines span the three-dimensional space of the sphere. It is apparent that a minimum of four lines is necessary to assure that there is no $\phi$- or $\phi'$- point. Furthermore, if a cut-set of more than four lines has no such point a subset of exactly four of them can be found that also has that property.

The result above could be stated more formally in terms of the language of convexity theory, with heavy reliance on Carathéodory's Theorem. This theorem states that if S is a set of points in n-dimensional Euclidean space and if P is a point of the convex hull of S then P is in the convex hull of some set consisting of n+1 or fewer points of S. The convex hull of a set is the smallest convex set that contains the given set. Alternately it is the set of points of the form $\Sigma\alpha_i P_i$ for points $P_i$ of the set and non-negative weights $\alpha_i$ with property $\Sigma\alpha_i = 1$.

For the special case in which all the picture lines pass through a common point the appropriate statement is that no $\phi$- or $\phi'$- point exists if and only if positive linear combinations of the associated normal vectors span the *two*-dimensional space associated with their common great circle. In this special case three (rather than four) vectors are necessary and sufficient. If more than three vectors are in the set and if their positive linear combinations span the associated two-dimensional space then some subset of exactly *three* vectors can be found that have that same property.

## THE BASIC REALIZABLE CUT-SET OF FOUR LINES

For the second part of our proof (that non-existence of a $\phi$- or $\phi'$- point implies realizability of the cut-set) we need to show that a four line cut-set having no such point is realizable. It is easy to show that four oriented picture lines having this property must necessarily form the patttern illustrated in Figure 7-a (or that pattern with all orientations reversed).

Since each oriented line in this representation has two different interpretations (a "+" line entering the region associated with the cut-set from that direction or a "-" line entering the cut-set from the opposite direction) there are sixteen different picture cut-sets to examine. One of these is shown in Figure 7-b. A possible associated trace is given in Figure 7-c. The four lines of that trace are of course at right angles to the corresponding picture lines, but their relative lengths are, as yet, undetermined.

The rates of change of w along the lines constituting the trace are equal to the distances from the picture plane origin to the four picture lines. By choosing the origin at the point indicated in Figure 7-b two of the rates of change of w can be made equal to zero. The other two rates of change are then determined and have some fixed ratio. By adjusting the lengths of the segments of trace between A and B and between A and D appropriately it is possible to guarantee that the net change in w around the trace is zero. Consequently there is at least one interpre-

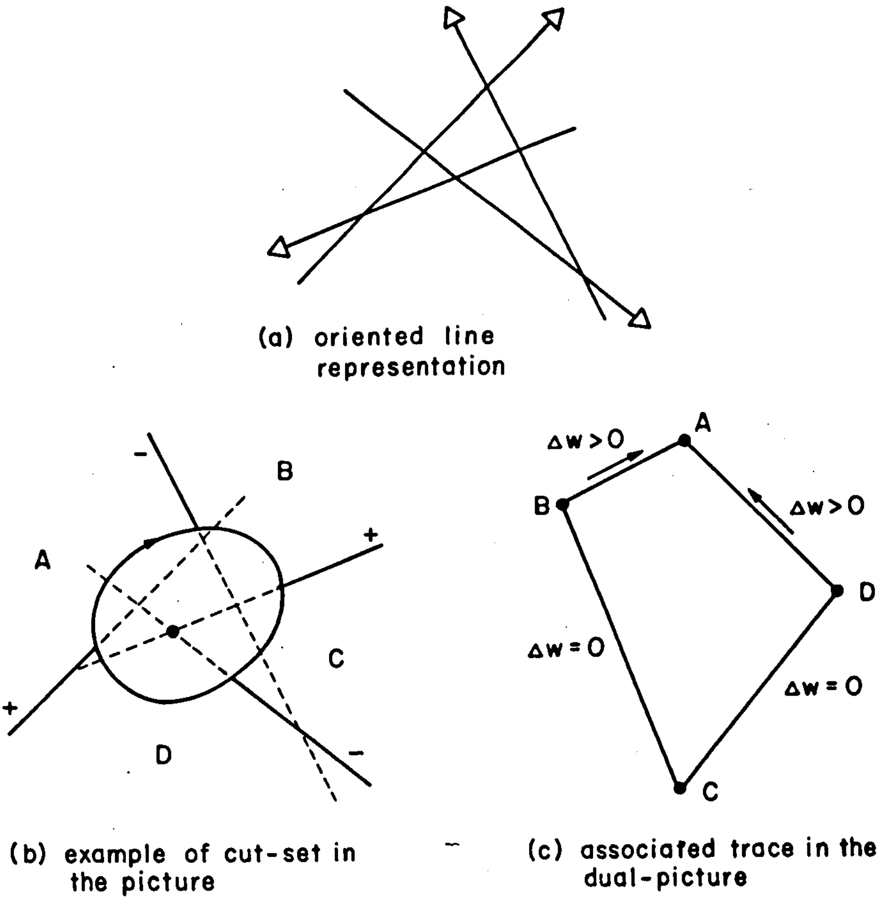



FIG. 7. Example of the basic realizable cut-set of four lines.

tation of the picture lines that demonstrates that the cut-set is realizable. Once we have such an interpretation in mind the location of the origin of the picture plane cannot influence the issue of realizability.

Each of the sixteen cut-sets associated with the oriented line representation of Figure 7-a can also be proved to be realizable in the way illustrated above. Another method for demonstrating realizability is to show explicitly some way of continuing and connecting the cut-set lines inside the picture contour. For our example there are two simple ways of accomplishing this (see Figure 8). Each of the other fifteen possibilities has at least one such simple completion. We have thus completed the second part of our proof for the case of four cut-set lines in general position.

For the special case of a cut-set of three lines that all pass through some common point the proof is even easier, and is left to the reader. The four trihedral configurations of lines that have only "+" or "-" labels in the catalog of

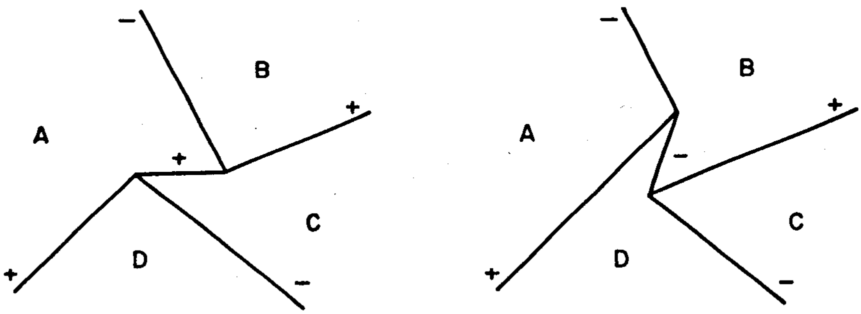



FIG. 8. Examples of completions of the cut-set of Figure 7b.

Figure 1 all correspond to this special case. All are equivalent (as can be seen by the line reversal technique mentioned earlier) to the Y-configuration with all lines labelled "+".

## COMPLETION OF THE PROOF

To complete our proof that lack of a $\phi$- (or $\phi'$-) point implies realizability of a cut-set we need to show that lines can be added to the basic four-line cut-set so that the augmented cut-set is also realizable. We shall give a brief presentation of the essential idea. Consider the realizable cut-set and associated closed trace of Figure 7-b and 7-c. The four components of the trace can be considered to be four vectors that sum to zero and that span the three-dimensional space (u,v,w) of the dual-scene. The addition of some other line to the cut-set in the picture corresponds to adding another vector component to the trace. The direction taken by this new vector component is determined by the orientation of the picture line and its distance from the picture origin. The length of the new component corresponds to the dihedral angle that is associated with the new picture line. If that dihedral angle is zero the new cut-set is obviously realizable.

If that dihedral angle is not zero but has some magnitude that is small compared with the other dihedral angles associated with the original four-line cut-set those angles may be adjusted slightly to compensate for the new line. Although this fact is not obvious when we consider only the picture, it is apparent when we examine the associated dual-scene. Because the original four vector components span the (u,v,w) space the new vector component can be expressed as a positive linear combination of those four vectors. That is, the sum of the new (fifth) vector and a modified positive linear combination of the original four can be made equal to zero. Therefore the trace can be made to close, as it did originally. Addition of other lines to the cut-set can be compensated for in the same way.

A similar argument holds if we add one or more new lines to any one of the basic cut-sets of three lines in the special case in which all these lines are incident at a common point.

We conclude that any cut-set of picture lines (that are labelled with "+" or

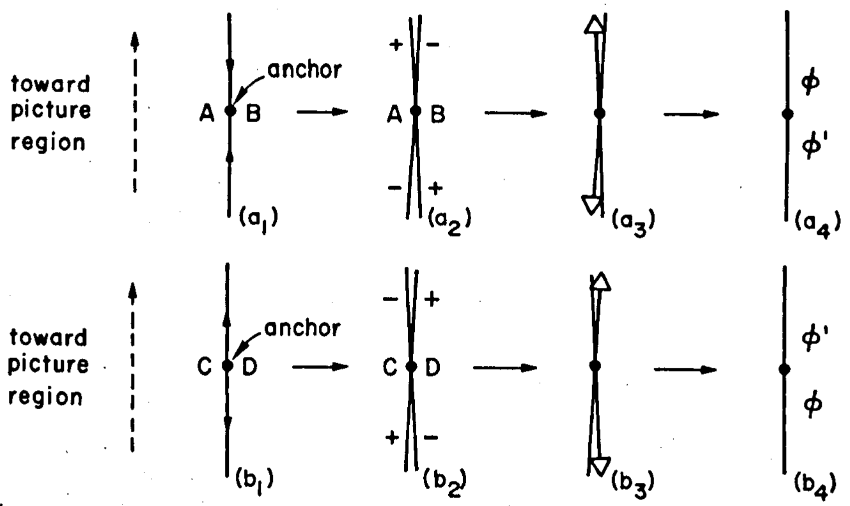



FIG. 9. Picture equivalents for arrow-labelled lines.

"-" labels only) is realizable if and only if the trace can be made to close in the dual-scene. Furthermore, we have seen that this condition is equivalent to the *non*-existence of a $\phi$-point or $\phi'$-point in the picture plane.

## CUT-SETS THAT CONTAIN ARROW-LABELLED LINES

In order to determine the realizability of a cut-set containing arrow-labelled lines we shall replace each such line by a pair of lines, one labelled "+" and the other labelled "-". We recall that an arrow label on a picture line means that the plane to the right of the arrow is closer to the viewer than the region represented to the left of the arrow. This region to the left may be the background against which the other plane is seen or it may simply be a more distant plane surface, perhaps one on another polyhedron.

Consider now the special case depicted in Figure 9-a$_1$. We consider the two planes A and B, having a common point at the position referred to as the "anchor" at which point the appropriate direction for the arrow label changes. In order for both arrow labels to be appropriate it might be, for example, that the A-plane has a "southward" component of tilt and the B-plane has a "northward" component of tilt. The configuration shown is a most unusual one and would require a special vantage point for the camera, one that would make it appear that the boundary of the A-plane represented above the anchor was an extension of the boundary of the B-plane represented below the anchor.

One physical situation that could be approximated by that representation is that shown in Figure 9-a$_2$ in which we assume that the angle between the two picture lines is arbitrarily small. The two small sectors could then be thought of as "cliff faces" seen nearly edgewise. If we assume that the direction to the picture region associated with the cut-set is up, as indicated in the figure, the

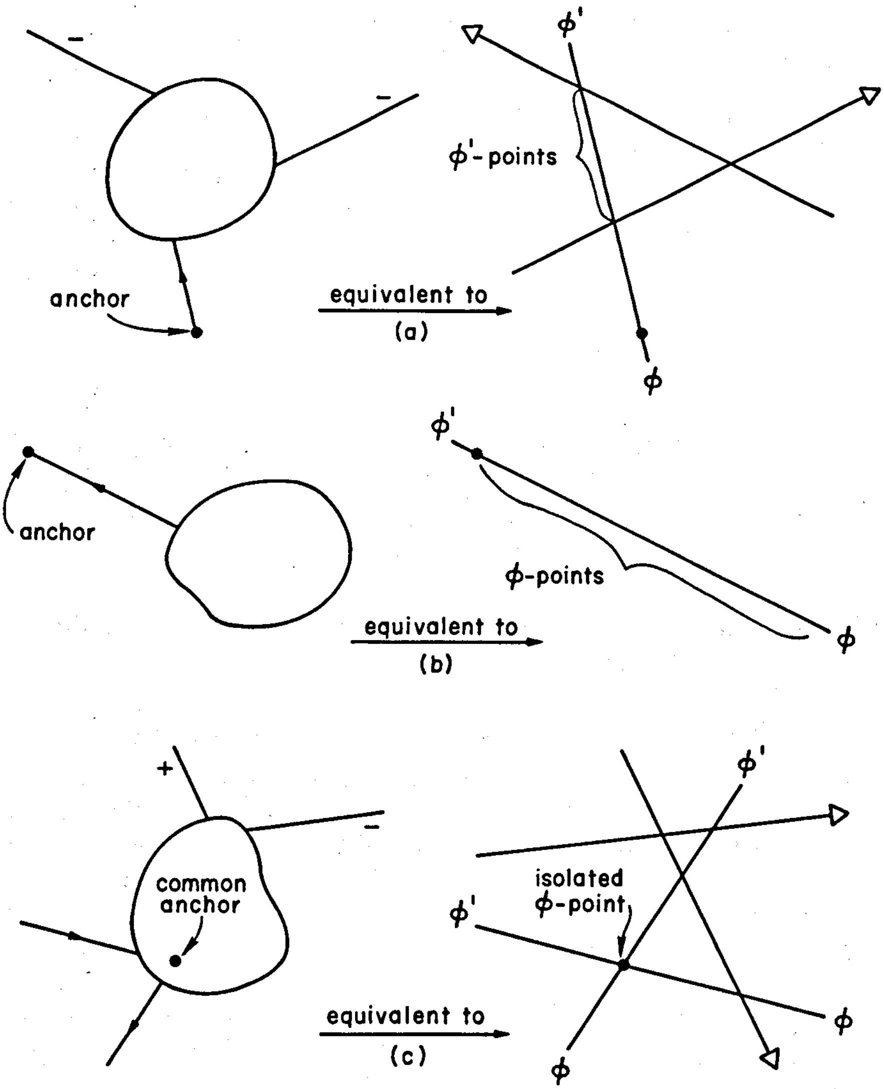



FIG. 10. Three examples of unrealizable cut-sets.

corresponding oriented line representation would be that of Figure 9-$a_3$. In the limit (as the angle between the two lines approaches zero) we see that the only possible location for $\phi$-points is on the line above the anchor; the only possible location for $\phi'$-points is on the line below the anchor. The equivalent representation in Figure 9-$a_4$ indicates those facts. Another special case, shown in Figure 9-b, establishes an equivalent for a related situation in which the plane C is tilted north and the plane D is tilted south.

In order to determine the realizability of a cut-set containing one or more arrow-labelled lines each such line is replaced by its equivalent (either from

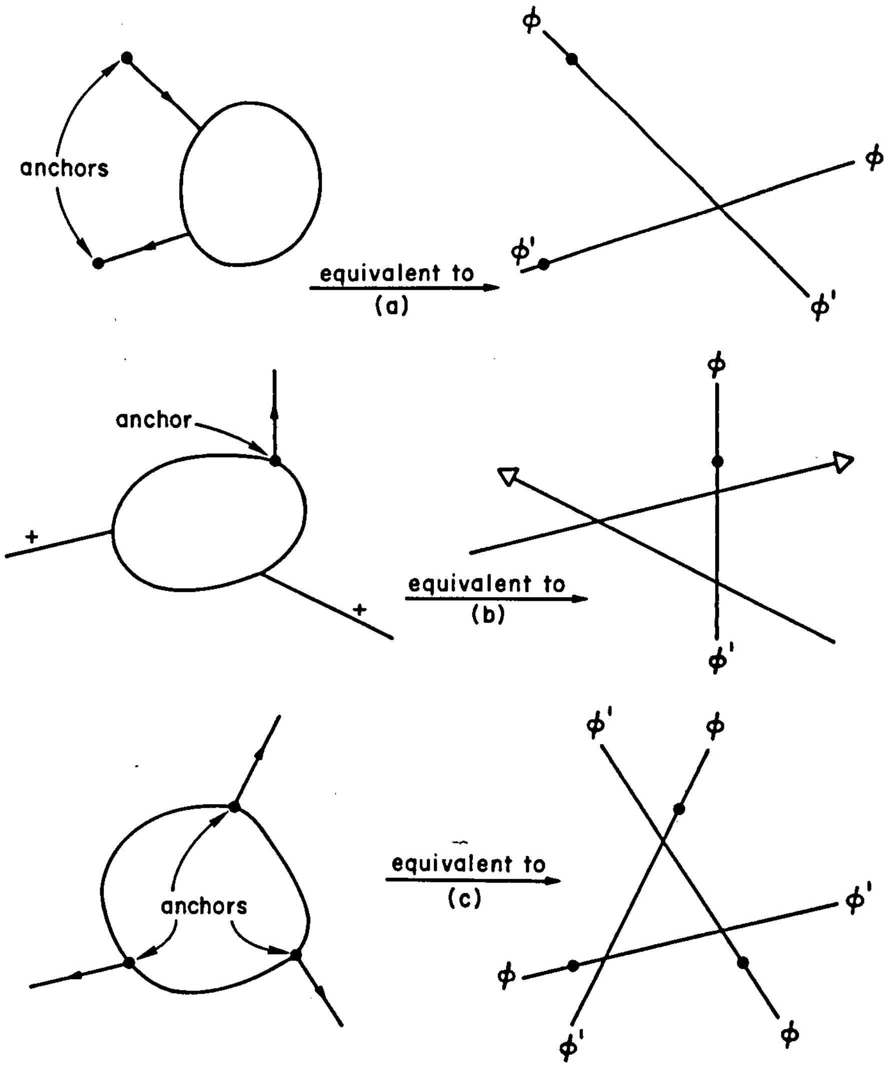



FIG. 11. Three examples of realizable cut-sets.

Figure 9-a4 or 9-b4). Any possible $\phi$- (or $\phi'$-) point must lie *on* the corresponding half-lines of all such equivalents as well as being to the right of (or the left of) all oriented equivalents to "+" or "-" labelled lines. Figures 10 and 11 give some simple examples of both realizable and unrealizable cut-sets.

In general, the position of the "anchor" on an arrow-labelled line may have to be known before the realizability of the cut-set containing that line can be determined. If the region to the left of an arrow-label is an infinitely remote background the placement of the anchor on the line may be chosen arbitrarily. In that case, however, the locations of possible $\phi$- (or $\phi'$-) points for the picture are still constrained to be only on that line.

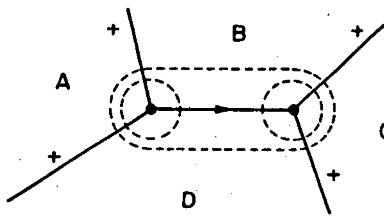



FIG. 12. A doubly-anchored arrow-labelled line.

Special cases of realizable cut-sets that contain arrow-labelled lines are included in the catalog of Figure 1. It is easy to prove that any single-node cut-set that contains one of the configurations of that catalog is also realizable.

A special case worthy of attention is that of a configuration of lines that constitutes a "T". In the restricted trihedral language (Huffman, 1971) this configuration can only represent one scene edge obscuring another. Thus in that language the bar of the "T" should always be assigned a left-pointing arrow-label. The node at the junction of the three line segments does not correspond to a vertex of a polyhedron in that case.

When arbitrary numbers of planes can be incident at a common polyhedral vertex a "T" can have many other interpretations. This configuration requires special treatment because two of the components of the configuration lie along exactly the same line. In this paper there is included in the Appendix a catalog and brief summary of results pertaining to the interpretations that a "T" may have.

## GENERAL CONDITIONS FOR REALIZABILITY OF A PICTURE

It would be tempting to speculate that if all cut-sets of a picture are realizable then the picture itself is. That unfortunately is not the case. Consider, for instance, the portion of picture shown in Figure 12. The cut-sets associated with both the left and right picture nodes are each easily proved to be realizable (each contains a subset of labelled lines contained in the catalog of Figure 1). It is also easy to prove that the larger cut-set assoicated with the pair of nodes is realizable (it is an example of the basic four-line cut-set discussed earlier). By examining the left node we can see that the planes A, B, and D would all have the associated vertex in common. Similarly we see that the planes B, C, and D all would contain the other vertex. Thus there is evidence that the arrow-labelled line is anchored at each of those two points. This is, of course, ridiculous. The plane D cannot be nearer the viewer along the line than plane B is and at the same time have two points in common with plane B.

Another more complicated example of a portion of picture that is not realizable is shown in Figure 13. There are seven ways of enclosing subsets of the three nodes and thus seven cut-sets, each of which is itself realizable. The one associated with all three nodes is effectively the same as in the third example in Figure 11. The others are left as exercises for the reader.

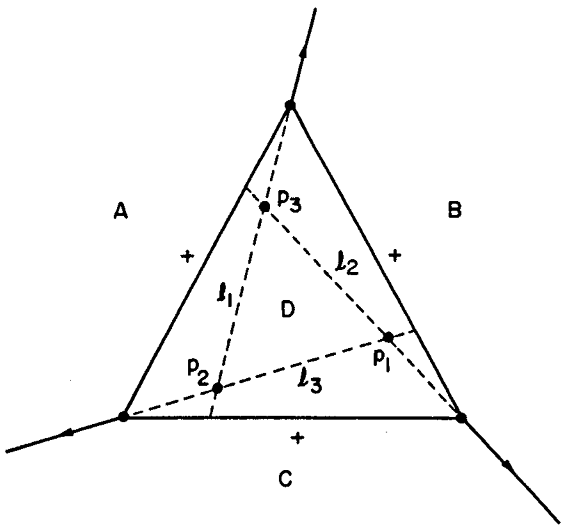



FIG. 13. An unrealizable portion of a picture that has all cut-sets realizable.

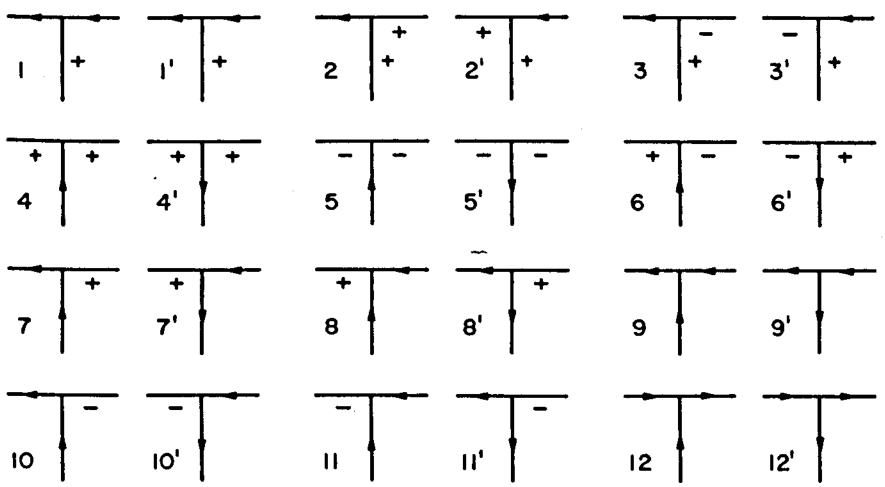



FIG. 14. Realizable T-configurations requiring no special vantage point.

The fact that the picture is nevertheless not realizable can be demonstrated as follows. We note that the anchor (the top picture node) on the arrow-labelled line implies that on the extension of this line on the other side of the anchor the B-plane would be further away from the viewer than would the A-plane. Thus along the segment $\ell_1$ we have $z_A < z_B$. Similarly, along $\ell_2$ we have $z_B < z_C$ and along $\ell_3$ we have $z_C < z_A$. Therefore at $p_1$ (the point common to $\ell_2$ and $\ell_3$) we have $z_B < z_A$. Therefore somewhere between $p_1$ and $\ell_1$ is the locus (a straight line) along which $z_A = z_B$. Similarly between $p_2$ and $\ell_2$ is a line along which $z_B = z_C$ and between $p_3$ and $\ell_3$ is a line along which $z_C = z_A$. We conclude that

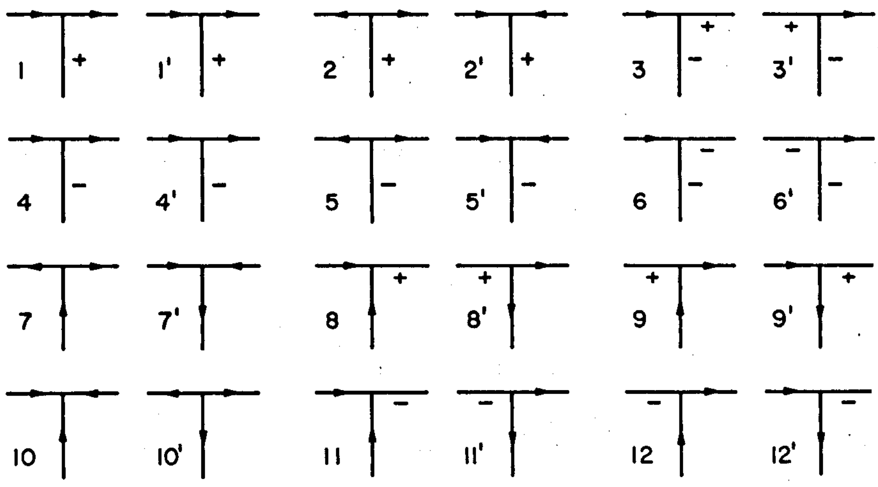



FIG. 15. Realizable T-configurations requiring a special vantage point.

somewhere inside the triangle bounded by the dotted lines is a picture point $p_0$ at which $z_A = z_B = z_C$.

Imagine now a line drawn between $p_0$ and the top picture node. That construction line would obviously have to lie to the left of (counterclockwise from) the arrow-labelled line through that top node. Along that construction line the A- and B-planes if extended would intersect and that line of intersection would be convex (be associated with a "+" label) as seen from the vantage point of the viewer. We would, in turn, conclude that the region in which the topmost arrow-labelled line lies is one for which $z_A < z_B$. But this conclusion is in contradiction with $z_A > z_B$ which is implied by the arrow label on the line itself. The contradiction is apparent and we conclude that the picture is not realizable.

## SUMMARY

We conclude that the realizability of all cut-sets of a picture, the lines of which have been tentatively labelled, is a necessary but not sufficient condition for the realizability of the picture with those "meanings" associated with the lines. The tests for realizability developed in this paper nevertheless are easy to apply to the various cut-sets and do, in effect, generalize as far as is possible the catalog of trihedral configurations (see Figure 1) derived earlier (Huffman, 1971).

It seems likely to the author that labelled pictures that pass the cut-set tests developed in this paper but are still not realizable (see, for instance, Figures 12 and 13) will, as a practical matter, have to be tested for realizability by simply attempting to find a three-dimensional dual representation (see Huffman, 1976) consistent with the picture. This is, of course, an exceptionally weak claim since it is equivalent to saying that a picture is realizable if and only if it is possible to determine locations of planes in the scene that intersect and obscure each other in ways that yield the lines of the given picture.

## APPENDIX: CATALOG OF REALIZABLE T-CONFIGURATIONS

In a polyhedral picture a "T" is an example of a configuration in which two or more lines have interdependent directions. In the case of a "T" the pair of lines constituting the top crossbar have exactly the same direction. Perhaps the simplest interpretation of a "T" is that the crossbar depicts an edge that obscures the edge represented by the third line. In that case the junction of the "T" does not represent a vertex of the scene. If that junction does represent a polyhedral vertex than many other interpretations are possible, all requiring more than three planes at that vertex.

Since there are four possible line labels there are $4^3 = 64$ ways of labelling the three lines of a "T". Nineteen of these are not realizable. Of the remaining 45 that are realizable there are 23 (see Figure 14) that have the property that the two components of the crossbar represent a single line in the three-dimensional scene. These 23 configurations thus do not change their basic form when photographed from slightly different camera positions. We note that the "-" label is never appropriate on the third line of the "T". The remaining 22 labelled configurations are realizable but would require a special vantage point for the camera in order for the picture to show the two crossbar components exactly aligned.

The entries of Figures 14 and 15 are arranged in pairs each member of which is a mirror-image of the other. When the mirror image of a scene is labelled one should keep the "+" and "-" labels of the resulting picture the same. The directions of arrow-labels on the remaining lines should be these obtained by reversing the directions on the arrow-labels when the original line-labelled picture is viewed in a mirror. Some labelled configurations (No. 1 in Figure 14 and No. 1 and No. 4 in Figure 15) remain invariant under this transformation.

Three of the catalog entries (No. 12 and No. 12′ in Figure 14 and No. 1 in Figure 15) have the additional property that they are not realizable unless the 180° sector above the crossbar represents a distant "background". That sector cannot represent a plane that contains the indicated vertex as one of its points.

## REFERENCES

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